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APPLICATIONS OF STATISTICAL DISTRIBUTIONS IN ACOUSTIC MODEL EVALUATION

Log-gamma probability density and distribution curves are presented as graphical aids in assessing the statistical character of propagation loss data

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RW McGirr

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NAVAL OCEAN SYSTEMS CENTER
SAN DIEGO, CALIFORNIA 92152

NOSC

NAVAL OCEAN SYSTEMS CENTER, SAN DIEGO, CA 92152

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Technical Director

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Computer graphics have been incorporated into a model evaluation software package that is intended to provide insight into the statistics of propagation loss. Specifically, the user may optionally superimpose plots of theoretical distribution curves on empirical distribution displays. A tutorial review of the gamma distribution is presented in the context of multipath fluctuations, followed by a comprehensive discussion of the log-transformed gamma distribution. An example is included to illustrate the application of the maximum likelihood method of parameter estimation.

CONTENTS

1. INTRODUCTION

The purpose of this document is to describe the statistical distribution models that have been adopted for use in a model evaluation demonstration software package developed for the Acoustic Model Evaluation Committee (AMEC). Most of the statistical tests and measures of accuracy that have been incorporated into this package are documented elsewhere (see references in McGirr [1979]). However, as an aid in assessing the statistical nature of empirical data, a means has been provided to allow graphical comparison against curves generated by theoretical distribution models. For example, the empirical probability density function (pdf) of data derived from propagation loss measurements can be compared graphically against curves generated by the log-gamma pdf. As another example, residual errors obtained by taking differences between propagation loss measurements and predictions can be compared graphically against curves generated by a scaled Student's t pdf. Also, Kolmogorov-Smirnov tests have been included in the software package to allow testing empirical cumulative distribution functions (CDF) against either the normal (or Gaussian) CDF or the log-gamma CDF. Certainly the popularity enjoyed by both the normal and the Student's t distributions precludes the necessity of elaborate discussion concerning either one. The log-gamma distribution, on the other hand, does not accede to general application as do the normal and the Student's t. What is more, the log transformation adopted here is reversed in sign from that typically cited in the acoustics literature, and consequently expanded discussion is desirable.

Section 2 discusses the impetus for choosing the gamma distribution vice either the exponential distribution or the "sum-of-exponentials" distribution. Maximum likelihood parameter estimates are discussed as an alternative to estimates based on the method of moments. Some basic properties of the gamma

distribution are reviewed, and one of the parameters of the gamma distribution is given an interpretation based on a comparison with the distribution of the sum of two exponentially distributed variables.

Section 3 discusses the log-gamma distribution that is obtained by applying a log-transformation to gamma distributed intensities. The particular transformation applied here is reversed in sign from that usually reported in the acoustics literature, and an explanation for this reversal is provided. Basic properties of the log-gamma density and distribution functions are presented, and a brief discussion of parameter estimation is also included.

The report concludes with a few remarks pertaining to useful extensions of the AMEC software package, and possible applications of the log-gamma and other distribution functions to the problem of placing bounds on propagation loss predictions.

2. THE GAMMA MODEL

Statistical models of intensity fluctuations for signals (and noise) propagating through an ocean medium have been presented and analyzed by Dyer [1970, 1973]. More recent articles by Mikhalevsky and Dyer [1978] and by Mikhalevsky [1979] have expanded the distribution theory pertaining to signal and noise statistics under various conditions. Of particular interest here are probability density and distribution functions that are useful in characterizing the statistical nature of propagation data.

For harmonic signals under well-developed multipathing, Dyer [1970] determines the probability density function (pdf) of intensity (short-time average of mean square pressure) to be exponential (Rayleigh amplitude). Letting x denote intensity, then x has pdf

$$p(x) = \lambda^{-1} \exp(-x/\lambda), \ x>0, \ \lambda>0, \qquad 2.1$$

with mean $\mu_x = \lambda$ and variance $\sigma_x^2 = \lambda^2$. Coming from a radically different point of view, Dozier and Tappert [1978] develop a statistical theory of normal mode amplitudes in a random ocean. For acoustic frequencies less than about 250 Hz and under saturation conditions, they arrive at the same pdf in the limit as the number of modes increases without bound.

When the multipath intensities are not identically distributed but can be sorted into M groups, then each x_m, $m = 1,2,\ldots,M$, is exponentially distributed with mean λ_m. The pdf of the sum $x = \sum_{m=1}^{M} x_m$, say f(x), takes a form resembling a weighted sum of exponentials [Dyer, 1973]. That is,

$$f(x) = \sum_{m=1}^{M} (w_m/\lambda_m)\exp(-x/\lambda_m) \qquad 2.2$$

with mean $\mu_x = \sum_{m=1}^{M} \lambda_m$, variance $\sigma_x^2 = \sum_{m=1}^{M} \lambda_m^2$, and "weighting factors" given by

$$w_m^{-1} = \prod_{i \neq m}^{M} (1 - \lambda_j/\lambda_m).$$

On the other hand, if the $\lambda_m = \lambda$ for all m, then the sum has the gamma pdf, say

$$g(x) = \frac{x^{M-1} e^{-x/\lambda}}{\lambda^M \Gamma(M)} \qquad 2.3$$

with mean $\mu_x = M\lambda$ and variance $\sigma_x^2 = M\lambda^2$ (see for example p 126 of Mood and Graybill [1963]).

Allowing for distinct group means has intuitive appeal, and when the underlying assumptions prevail the pdf f(x) is probably a more realistic model than the gamma pdf. There is an unappealing aspect of this model, however, which stems from the less than satisfactory means available to estimate the set of λ_m from data. Ambiguous estimates of the λ_m may be obtained from

$$\Sigma\lambda_m = \bar{x} = \frac{1}{N} \Sigma x_n.$$

Thus, unless there exists a priori information identifying M-1 of the λ_m, additional information must be acquired from higher-order sample moments. Procedures based on higher-order moments are computationally prohibitive for M>2 due to the range of values typical of intensities.

For the special case of only two groups (M = 2) the method of moments becomes less impractical, but such a method, though simple conceptually, is not without drawbacks. Consistent estimates of λ_1 and λ_2 may be obtained from (see p 270 of Lindgren [1978])

$$\lambda_1 + \lambda_2 = \bar{x}$$

and $\lambda_1^2 + \lambda_2^2 = S^2$

where $\bar{x}$ and S^2 are the sample mean and variance.

If $\lambda_1 > \lambda_2$, say, then

$$\lambda_1 = \tfrac{1}{2}(\bar{x} + \sqrt{2S^2 - \bar{x}^2})$$

and $\lambda_2 = \tfrac{1}{2}(\bar{x} - \sqrt{2S^2 - \bar{x}^2})$

Since the λ_m are real and positive, the sample variance is necessarily constrained to lie in the interval $(\bar{x}^2/2, \bar{x}^2)$. The likelihood that this constraint could be easily violated suggests that the "sum of exponentials" model lacks the robustness necessary to tolerate possible discrepancies between the model's underlying assumptions and reality.

2.1 The Gamma as a Generalized Exponential

Even if a reasonable technique to estimate the λ_m could be devised, there is still no clear cut way to determine M, except in "controlled" experiments. Thus there is ample incentive to consider the gamma pdf expressed in the form

$$g(x) = \frac{x^{v+1} e^{-x/\lambda}}{\lambda^v \Gamma(v)} \qquad 2.4$$

as an appropriate density. Note that M has been replaced by v. The purpose of this exchange is to emphasize that this parameter is no longer restricted to integer values. An interpretation of v is given in Section 2.4.

The idea of using the gamma pdf as a generalized exponential is not new. In his development of the statistics of energy fluctuations, Rice [1954] suggests the gamma pdf expressed in the form

$$p(E) = a^{n+1} E^n e^{-aE}/\Gamma(n+1)$$

as an appropriate density of

$$E = \int I^2(t)dt.$$

I(t) is the instantaneous thermal noise current and the integration extends over a finite time interval. The reader should have no difficulties in making the associations: x with E, v with n + 1, and λ with a^{-1}. To obtain estimates of the parameters a and n, Rice employs the method of moments where the sample moments are derived from the spectral density function of the process. Two approaches to the parameter estimation problem are presented in the next section.

2.2 Parameter Estimation

2.2.1 Maximum Likelihood Estimates

One of the advantages of the gamma distribution is that estimates of λ and v may be obtained from first-order statistics. For a sample of size N the likelihood function takes the form

$$L(\lambda,v) = \prod_{n=1}^{N} x_n^{v-1} e^{-N\bar{x}/\lambda}/(\lambda^v\Gamma(v))^N,$$

where $\bar{x}$ is the arithmetic sample mean

$$\bar{x} = \frac{1}{N}\sum_{n=1}^{N} x_n.$$

If $\tilde{x}$ denotes geometric sample mean, that is

$$\tilde{x} = (\Pi x_n)^{1/N},$$

then

$$\ell n L = N[(v-1)\ell n\tilde{x} - \bar{x}/\lambda - v\ell n\lambda - \ell n\Gamma(v)].$$

Maximum likelihood estimates of λ and v are obtained by solving the likelihood equations

$$\frac{\partial \ell n L}{\partial \lambda} = N(\bar{x}/\lambda^2 - v/\lambda) = 0$$

$$\frac{\partial \ell n L}{\partial v} = N(\ell n\tilde{x} - \ell n\lambda - \psi(v)) = 0,$$

where $\psi(v) = d\ \ell n\Gamma(v)/dv$ is the digamma (or psi) function. Thus the pair of equations

$$v\lambda = \bar{x} \qquad 2.5$$

and

$$\ell nv - \psi(v) = \ell n(\bar{x}/\tilde{x}) \qquad 2.6$$

provide an iterative scheme to obtain estimates of both λ and v from first-order sample statistics.

This method of parameter estimation offers a computational advantage over methods using second-order moments. The only moment that need be calculated in intensity space is $\bar{x}$. The geometric mean is actually calculated in dB space as an arithmetic mean. That is, under the transformation $y = a\ \ell n(x/v\lambda)$, where $a = -10 \log e$,

$$\ell n\tilde{x} = \ell nv\lambda + |a|^{-1} \frac{1}{N} \Sigma\, y_n$$
$$= \ell n\bar{x} + |a|^{-1} \frac{1}{N} \Sigma\, y_n$$

and hence

$$\ell nv - \psi(v) = |a|^{-1} \frac{1}{N}\Sigma y_n = \bar{y}/|a|. \qquad 2.7$$

At first glance this expression does not appear to offer a straightforward approach to the problem of estimating v. However, since the left hand side of this equation is independent of sample data, it may be plotted against

v, thus providing a simple graphical procedure. This procedure is demonstrated in section 3.5.

2.2.2 Method of Moments

An alternative procedure is offered by the so-called "method of moments" (see p 278 of Lindgren [1978]), wherein μ_x and σ_x^2 are expressed in terms of the parameters v and λ and then equated to the corresponding sample statistics. Thus,

$$\mu_x = v\lambda = \bar{x},$$

and

$$\sigma_x^2 = v\lambda^2 = S^2,$$

from which

$$v = \bar{x}^2/S^2. \qquad 2.8$$

Rice [1954] suggests similar estimates, except instead of using the usual sample statistics, $\bar{x}$ and S^2, he expresses the moments in terms of the spectral density function of the process. More recently Frisk [1978] demonstrates the use of this method in estimating a parameter analogous to v for the so-called M-distribution (which may be obtained from the gamma by making the transformation $z = x/v$).

The maximum likelihood estimation procedure is generally preferred to other methods available. Lindgren notes, however, that the method of moments usually yields "efficient" estimates that serve as useful first-order approximations.

2.3 Some Properties of the Gamma Distribution

2.3.1 Gamma Density Curve

If $\lambda = 1$ the gamma pdf assumes the simple form

$$g(x) = \frac{x^{v-1}e^{-x}}{\Gamma(v)}, \; x>0, \; v>0.$$

A further simplification obtains when v is confined to the set of positive integers, say v = n, n = 1,2,..., so that

$$g_n(x) = \frac{x^{n-1}e^{-x}}{(n-1)!} \quad , \; x>0, \; n=1,2,3,\ldots \; . \qquad 2.9$$

Curves of this simplified form are illustrated in fig 1 for n = 1, 2, 3, and 4. For n = 1 the curve starts with a slope of -1, for n = 2, it starts with a slope of +1, a for $n > 2$ all curves have initial slope zero. The exponential curve (n = 1) has no interior points of inflection, the n = 2 curve has an inflection point at x = 2, and for $n > 2$ each curve has two points of inflection at $x = n - 1 \pm\sqrt{n - 1}$. The modes (or most probable values) occur at x = n - 1. When λ is not unity, the expected value of x is nλ and the mode occurs one λ-unit to the left at (n - 1)λ.

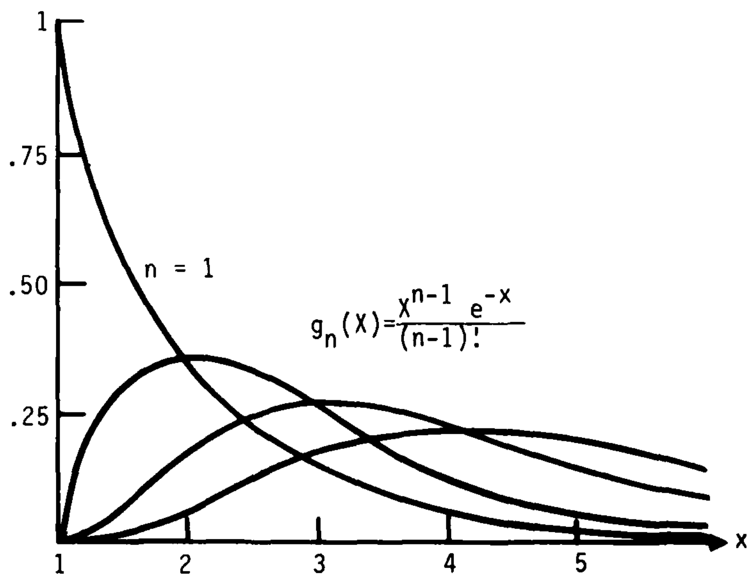

Figure 1. Gamma pdf curves for n = 1,2,3, and 4.

2.3.2 Moments

The moments of gamma variates may be obtained from

$$E\,x^n = \int_0^\infty \frac{x^{n+v-1}e^{-x/\lambda}dx}{\lambda^v\Gamma(v)} \tag{2.10}$$

Making the change of variable $z = x/\lambda$ yields

$$Ex^n = [\lambda^n/\Gamma(v)] \int_0^\infty z^{n+v-1}e^{-z}dz = \lambda^n\Gamma(n+v)/\Gamma(v) \tag{2.11}$$

Thus the first two moments are

$$Ex = v\lambda \qquad (\equiv\mu_x) \tag{2.12}$$

and

$$Ex^2 = v(v+1)\lambda^2 \tag{2.13}$$

from which the variance σ_x^2 is

$$\sigma_x^2 = E(x - \mu_x)^2 = Ex^2 - E^2x = v\lambda^2. \tag{2.14}$$

Higher-order central moments may be obtained from

$$E(x - \mu_x)^n = \int_0^\infty \frac{(x - \mu_x)^n x^{v-1}}{\lambda^v\Gamma(v)} e^{-x/\lambda}dx. \tag{2.15}$$

Making the same change of variable as above and simplifying yields

$$E(x - \mu_x)^n = \frac{1}{\Gamma(v)} \sum_{k=0}^{n} \binom{n}{k}\lambda^k(-\mu_x)^{n-k}\Gamma(v + k). \tag{2.16}$$

The third central moment, say m_3, is then

$$m_3 = 2v\lambda^3, \tag{2.17}$$

so that the coefficient of skewness, α_3 is given by

$$\alpha_3 = m_3/m_2^{3/2} = 2v^{-1/2}. \tag{2.18}$$

Similarly the fourth central moment is

$$m_4 = 3v(v + 2)\lambda^4 \qquad 2.19$$

and hence the coefficient of kurtosis, α_4, is given by

$$\alpha_4 = m_4/m_2^2 = 3\,\frac{v + 2}{v} \qquad 2.20$$

Note that as v gets large these coefficients tend toward Gaussian values, that is

$$\lim_{v \to \infty} \alpha_3 = 0 \qquad 2.21$$

and

$$\lim_{v \to \infty} \alpha_4 = 3 \qquad 2.22$$

The rates of approach to these limits are demonstrated in fig 2.

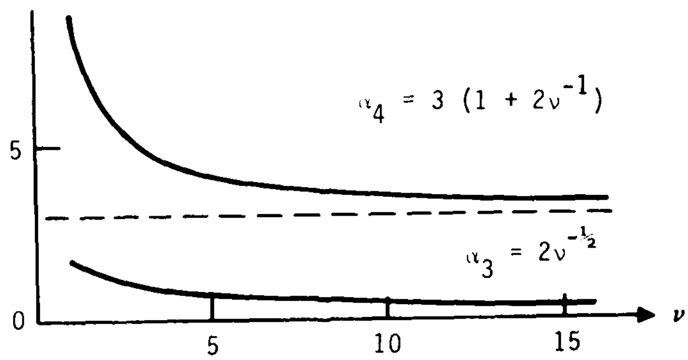

Figure 2. Convergence to Gaussian limits.

2.3.3 Gamma CDF

The gamma CDF may be expressed as

$$G_v(x) = \int_0^x \frac{t^{v-1}e^{-t/\lambda}}{\lambda^v \Gamma(v)}\, dt \qquad 2.23$$

which for arbitrary v is not generally expressable in closed form. Tables of the incomplete gamma function [Pearson, 1934] may be used, however, making use of relationships and tables in Abramowitz and Stegan (A&S)*[1965] may be more convenient. For integer v, say v = n, a finite-term series obtains, that is

$$G_n(x) = 1 - e_{n-1}(x/\lambda)e^{-x/\lambda}, \qquad 2.24$$

where

$$e_n(x) = \sum_{k=0}^{n} x^k/k! \quad \text{(A\&S, p 262).}$$

When n/2 is an integer, tables of the chi-square distribution may be used. Making the transformation $z = 2x/\lambda$ yields the pdf

$$p(z) = \frac{z^{n-1}e^{-z/2}}{2^n\Gamma(n)}, \qquad 2.25$$

so that replacing n by k/2 and z by χ^2 yields

$$p(\chi^2) = \frac{\chi^{k-2}e^{-\chi^2/2}}{2^{k/2}\Gamma(k/2)}, \qquad 2.26$$

the chi-square pdf with k "degrees of freedom." Thus,

$$G_n(x) = P_{\chi^2}(2x/\lambda|2n) = 1 - Q_{\chi^2}(2x/\lambda|2n) \qquad 2.27$$

where P_{χ^2} and Q_{χ^2} are defined on p 940 of A&S, and tabulations of Q_{χ^2} begin on p 978.

2.3.4 Remarks Concerning Numerical Evaluation

When v does not take an integer or half-integer value, then either tables must be interpolated or algorithms must be prepared for computer

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* Subsequent citings to this reference are abbreviated A&S.

implementation. The algorithms used in the AMEC demonstration package are based on series and asymptotic expansions that appear on p 262-3 of A&S. Now

$$G_v(x) = \int_0^x \frac{t^{v-1}e^{-t/\lambda}}{\lambda^v\Gamma(v)}\,dt = \int_0^{x/\lambda} \frac{t^{v-1}e^{-t}}{\Gamma(v)}\,dt = P(v,\ x/\lambda) \qquad 2.28$$

where $P(v,z) = \gamma(v,z)/\Gamma(v) = 1 - \Gamma(v,x)/\Gamma(v)$. The functions $P(v,z)$, $\gamma(v,z)$ and $\Gamma(v,x)$ represent different ways of expressing the incomplete gamma function, and are defined on p 260 of A&S. Expanding e^{-t} and integrating yields the convergent series (A&S, 262)

$$P(v,z) = \frac{z^v}{v\Gamma(v)}\left[1 - \frac{v}{v+1}\frac{z}{1!} + \frac{v}{v+2}\frac{z^2}{2!} - \ldots\right]. \qquad 2.29$$

Integrating by parts ($u = e^{-t}$, $dv = t^{v-1}dt$) yields the slightly faster series

$$P(v,z) = \frac{z^v e^{-z}}{v\Gamma(v)}\left[1 + \frac{z}{v+1} + \frac{z^2}{(v+1)(v+2)} + \ldots\right], \qquad 2.30$$

but one that is still slow for large z. The asymptotic expansion

$$P(v,z) = 1 - \frac{z^{v-1}e^{-z}}{\Gamma(v)}\left[1 + \frac{v-1}{z} + \frac{(v-1)(v-2)}{z^2} + \ldots\right] \text{ (A\&S, p. 263)} \qquad 2.31$$

obtained by integrating by parts ($u = t^{v-1}$, $dv = e^{-t}dt$) is divergent for all z and noninteger v, but for small v, say $v < 3$, and "large" z, say $z \gtrsim 4\lambda$, only a few terms are required for sufficient accuracy.

2.4 Relationship to Sum of Two Exponentials*

The purpose of this section is to provide an interpretation of v as it ranges through values between two successive integers. For propagation data v

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* This idea was suggested to me by Mr LK Arndt.

is expected to cluster about $v = 1$, or $v = 2$, corresponding to one or two primary groups of multipaths. When there are two groups and the energy is proportioned equally among them, their mean values are equal and $v = 2$ exactly. However, when their mean values differ then v takes on some value between 1 and 2. To see if v can be related in some way to the proportioning of energy among two groups of multipaths, it is compared with a parameter r introduced into the "sum of exponentials" pdf.

Let x_1 and x_2 represent the random variables associated with two groups of multipaths, then each has a pdf of the form

$$p_i(x) = \lambda_i^{-1} \exp(-x/\lambda_i), \; x > 0, \; i = 1,2, \qquad 2.32$$

where each x_i has mean λ_i. Without loss of generality let $\lambda_1 > \lambda_2$ and introduce the parameter r $(0<r<1)$ such that with $\lambda_1 = \lambda$ then $\lambda_2 = r\lambda$. The pdf of $x = x_1 + x_2$ is given by (for example see p 317 of Parzen [1960])

$$\begin{aligned} p(x) &= \int_0^x p_1(t)p_2(x-t)dt \\ &= \frac{\exp(-x/\lambda) - \exp(-x/r\lambda)}{\lambda(1-r)} \end{aligned} \qquad 2.33$$

and $E(x) = \lambda_1 + \lambda_2 = (1+r)\lambda$.

For gamma distributed variates, $E(x) = v\lambda$. Evidently then the parameter v corresponds to $1 + r$, at least to first order. This correspondence does not hold up when higher-order moments are compared, and exact correspondence obtains only at the limit points of r. Thus the implied linear relationship $v = r + 1$ is only approximate. Nevertheless, the interpretation is clear. For the case being considered, as v increases from 1 to 2 the partitioning of energy varies from complete imbalance (only one group) to complete balance (two groups with equal means).

The comparison at hand is illustrated schematically in figure 3.

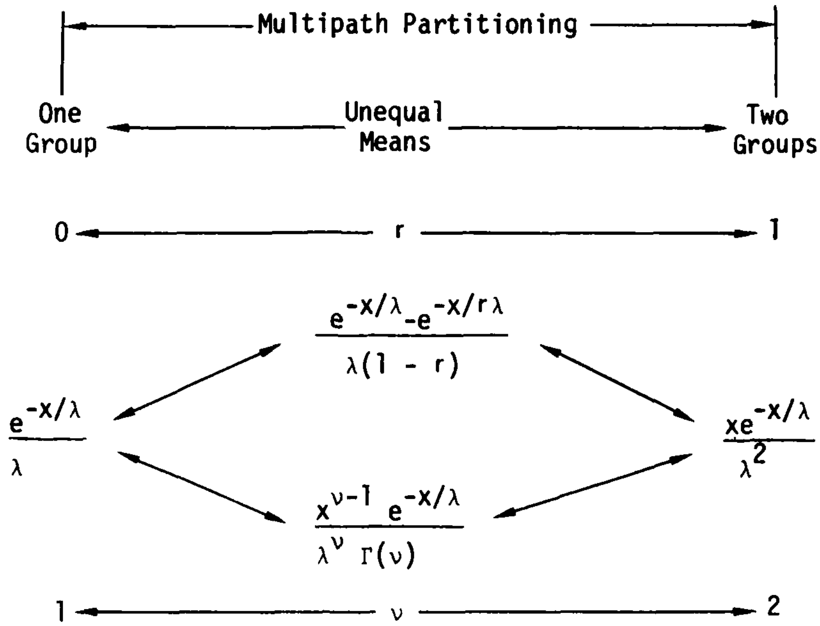


Figure 3. Correspondence between ν and r.

To see how well the gamma pdf approximates the sum of two exponentials, let $\lambda = 1$ and $r = 1/2$ so that the sum-of-exponentials pdf is

$$p(x) = \frac{e^{-x}-e^{-2x}}{1/2} = 2e^{-x}(1 - e^{-x}) \qquad 2.34$$

Under the assumed correspondence, the gamma pdf has $\lambda = 1$ and $\nu = 3/2$ so that

$$g(x) = x^{1/2}e^{-x}/\Gamma(3/2) = (2/\sqrt{\pi})\ x^{1/2}e^{-x} \qquad 2.35$$

These curves are illustrated in figure 4. The gamma pdf reaches a maximum ($x = .5$) slightly ahead of the other curve ($x \cong .69$), but the curves tend to coincide for $x > 2.5$. Indeed, at $x = 10$ $E(x) = 15$ the curves differ by only about 7×10^{-7}. Thus, although this discussion is necessarily brief and deals only with specifics, it demonstrates the basis for employing the simple gamma model with ν allowed to assume any positive real value as determined from a sample.

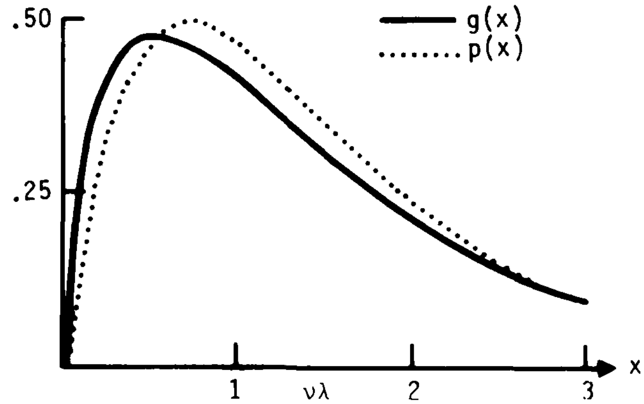


Figure 4. Comparison of gamma pdf with sum-of-2 exponentials.

3. THE LOG-GAMMA DISTRIBUTION

3.1 The Log Transformation

The range of intensity values typically encountered in ocean acoustics tends to be discouragingly prohibitive both computationally and graphically. Log transforming to decibels (dB) circumvents possible computational problems (eg, underflow) and certainly results in more convenient graphics (eg, 60 units vice 10^6). Thus, for example, propagation loss versus range plots typically display loss in dB as illustrated in figure 5.

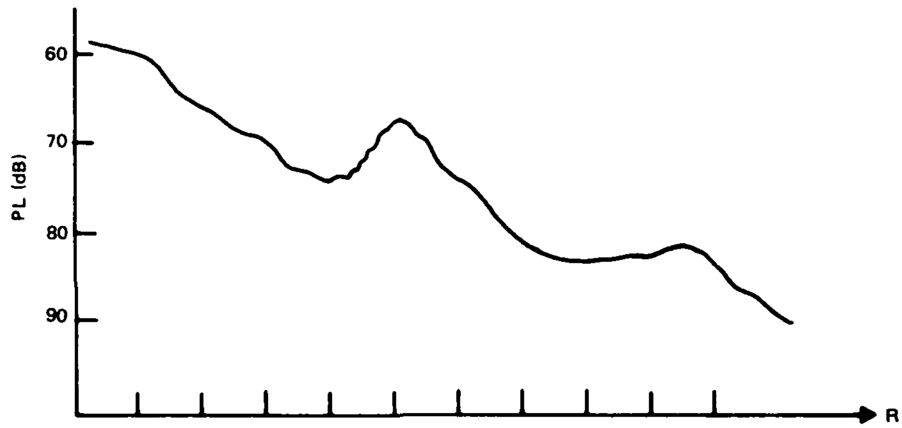


Figure 5. Plot format typical of propagation loss models.

The expression of acoustic quantities in dB is a well-established practice, so well that there is a natural inclination to transform mathematical "operations" from linear space to dB space as well. As Dyer [1970] points out, however, the average of log-transformed data does not equal the log of data averaged in linear space. For example, averaging in dB space vice averaging in linear space and then transforming can produce a 2.5 dB error. The reason for this discrepancy is made clear in section 3.3.

Proceeding to the transformation, let x denote intensity (actually the ratio of intensity at a given point to some reference intensity) then the loss H expressed in decibels is obtained by

$$H = -10 \log x.$$

For a sample of intensities x_n, $n = 1,2,\ldots,N$, obtained at ranges r_n the corresponding sample loss values are

$$H_n = H(r_n) = -10\log x_n, \qquad n = 1,2,\ldots,N.$$

If the range interval spanned by the r_n is not too large then "fluctuations about the mean" may be examined by calculating moments of $H_n - \langle H\rangle$, where $\langle H\rangle = -10\log\bar{x}$ and $\bar{x}$ is the arithmetic mean of the intensities.

Assuming that the x_n are independent random variables all deriving from a given population distribution, the corresponding distribution in dB space may be obtained through the transformation $H = -10 \log x$. The purpose of the minus sign is clarified in the next section. Mathematical manipulations are somewhat less cumbersome when the natural log is substituted for the common log. Hence the transformation is expressed as $H = a\,\ell n x$, where $a = -10 \log e$. Since the expected value of gamma distributed variates is $\nu\lambda$, the transformation of interest here takes the form

$$y = a\,\ell n(x/\nu\lambda). \tag{3.1}$$

Thus the gamma pdf

$$g(x) = \frac{x^{\nu-1}e^{-x/\lambda}}{\lambda^{\nu}\Gamma(\nu)}, \quad x > 0, \tag{3.2}$$

transforms to

$$f(y) = g(x(y))/|dy/dx|$$

$$= \frac{\nu^{\nu}}{|a|\Gamma(\nu)}\{\exp[y/a - \exp(y/a)]\}^{\nu}, \quad -\infty<y<\infty. \tag{3.3}$$

Despite its foreboding form, this pdf is not as exotic as its appearance might suggest. In fact it is routinely applied in the analysis of extreme-value problems.

Interestingly, if the motive for making a transformation were based on the desire to obtain a variate with constant variance, this same pdf (or one of similar form) would result. That is, suppose a transformation $z = h(x)$ is sought such that z has (approximately) constant variance. Expand h(x) about $x = \mu$ and assume that to a good approximation

$$h(x) \cong h(\mu) + h'(\mu)(x - \mu).$$

Then $E[h(x) - h(\mu)]^2 \cong [h'(\mu)]^2 \, E(x - \mu)^2$,

$$\text{or } \sigma_z{}^2 \cong [h'(\mu)]^2\sigma_x{}^2,$$

and if $\sigma_x{}^2$ can be expressed as some function of μ, say $V(\mu)$, then setting $h'(\mu) = [V(\mu)]^{-1/2}$ yields $\sigma_z{}^2 \cong 1$. Hence the desired transformation is provided by

$$z = h(x) = \int^{x} d\mu/\sqrt{V(\mu)}.$$

The variance of gamma distributed variates is $\nu\lambda^2$, so that for constant ν

$$h(x) \propto \int^{x} \frac{d\lambda}{\lambda} \propto \ell nx,$$

which implies that the transformation yielding a variable with constant variance is logarithmic. A discussion of this technique (referred to as the "angular transformation") may be found in Brownlee [1965].

3.2 Intensity-Loss Orientation

The material in this section is intended to orient the reader to the special application of the log-gamma distribution to the analysis of

propagation loss data as generally viewed by those engaged in propagation loss modeling. The majority of propagation loss models under consideration by the AMEC for evaluation offer an output option that generates a plot (or plots) of loss versus range. These plots are usually oriented like the one presented in figure 5, that is, with increasing values of loss directed downward. Since the gamma pdf is highly skewed, its orientation relative to such plots is important.

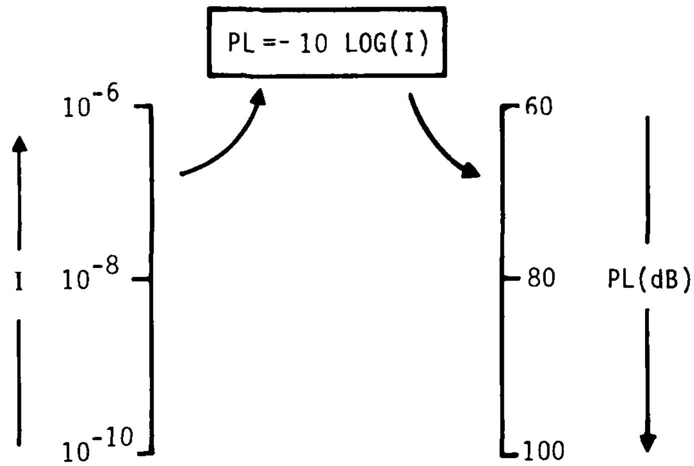
Figure 6 displays the orientation of distributions in linear (or intensity) space and in dB space, under the transformation $y = a\ell n(x/v\lambda)$ where $a = -10 \log e$. Values of x falling below $v\lambda$ (the mean) yield positive values of y (high loss) corresponding to the "right" tail of f(y). Similarly, values of x above $v\lambda$ yield negative values of y (low loss) corresponding to the "left" tail of f(y). Thus the random variable y is oriented in the same sense as propagation loss.

In terms of sample data,

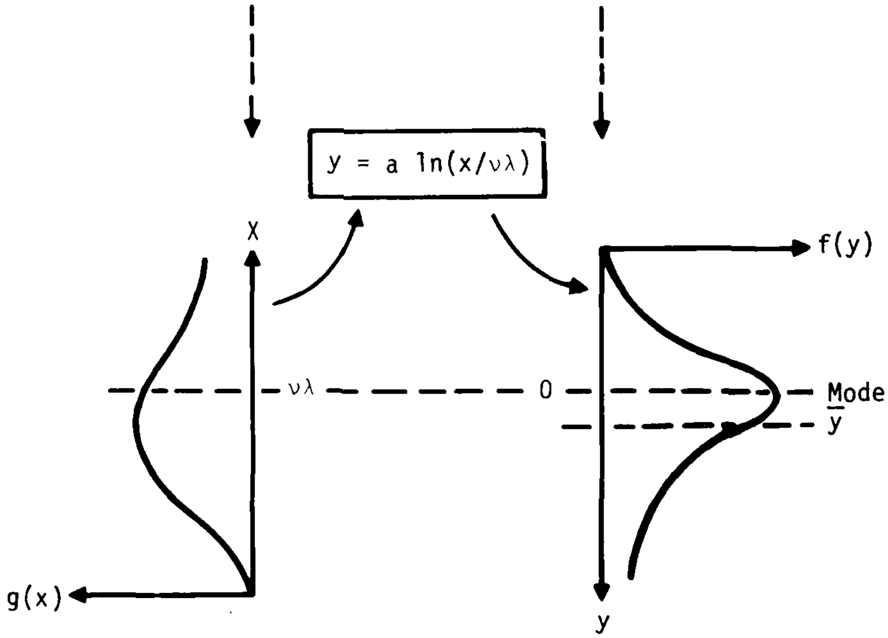
$$y_n = H_n - \langle H \rangle, \qquad 3.4$$

where $\langle H \rangle = -10 \log \bar{x}$ and $\bar{x}$ denotes the arithmetic mean of intensity data ($\bar{x} = \Sigma x_n/N$). Let $\bar{y}$ denote the mean calculated in dB space, then

$$\bar{y} = \frac{1}{N}\Sigma y_n = \frac{1}{N}\Sigma H_n - \langle H \rangle = \bar{H} - \langle H \rangle. \qquad 3.5$$



Transformation: Intensities to decibels



Transformation: g(x) to f(y)

Figure 6. Orientation of density curves g(x) and f(y) relative to intensity and decibel scales.

3.3 Moments

The gamma pdf

$$g(x) = \frac{x^{v-1}e^{-x/\lambda}}{\lambda^v\Gamma(v)} \;,\; x > 0$$

under the transformation $y = a\ \ell n(x/v\lambda)$ takes the form

$$f(y) = \frac{v^v}{|a|\Gamma(v)}\,(e^{y/a} - e^{y/a}) \;,\; -\infty<y<\infty. \qquad 3.6$$

The first moment of y is given by

$$Ey = \int_{-\infty}^{\infty} yf(y)dy$$

or, with the prospect of more tractable mathematics in x space,

$$Ey = E_x[a\ \ell n(x/v\lambda)]$$

$$= [a/\lambda^v\Gamma(v)] \int_0^{\infty} \ell n(x/v\lambda)x^{v-1}e^{-x/\lambda}\,dx \qquad 3.7$$

Making the change of variable $z = x/\lambda$ gives

$$Ey = [a/\Gamma(v)] \int_0^{\infty} \ell n(z/v)\ z^{v-1}e^{-z}dz$$

$$= [a/\Gamma(v)] \int_0^{\infty} (\ell nz - \ell nv)\ z^{v-1}e^{-z}dz$$

$$= [a/\Gamma(v)]\ \{\int_0^{\infty} \ell nz\ z^{v-1}e^{-z}dz - \ell nv \int_0^{\infty} z^{v-1}e^{-z}dz\}. \qquad 3.8$$

Now $\Gamma(v) = \int_0^{\infty} z^{v-1}e^{-z}dz,$ 3.9

and* $\dfrac{d\Gamma(v)}{dv} = \int_0^{\infty} \ell nz\ z^{v-1}e^{-z}dz,$ 3.10

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* I thank Mr LK Arndt for bringing this fact to my attention.

and hence

$$Ey = a[\Gamma'(v)/\Gamma(v) - \ln v]. \tag{3.11}$$

Since $\Gamma'(v)/\Gamma(v) = d\ln\Gamma(v)/dv \equiv \psi(v)$, the digamma (or psi) function (see eq 6.3.1, A&S), then the expression for the first moment of y simplifies to

$$Ey = a[\psi(v) - \ln v]. \tag{3.12}$$

Evaluating this expression at v = 1 yields (see eq 6.3.2, A&S)

$$\begin{aligned} Ey &= a\,\psi(1) \cong (-4.343)(-.5772) \\ &\cong 2.5 \text{ (dB)} \end{aligned} \tag{3.13}$$

which agrees with the result obtained by Dyer [1970] for the "log-exponential" pdf.

Let $\mu \equiv E(y)$ so that the higher-order central moments are given by

$$E(y - \mu)^n = \Gamma^{-1}(v) \int_0^\infty [a \ln(z/v) - \mu]^n z^{v-1} e^{-z} dz, \tag{3.14}$$

after making the same change of variable as above. Applying the binomial expansion to the quantity in brackets and integrating over the sum yields

$$E(y - \mu)^n = a^n \left[\sum_{k=0}^{n-1} \binom{n}{k} (-q)^k \int_0^\infty \frac{\ln^{n-k} z \; z^{v-1} e^{-z} dz}{\Gamma(v)} + (-q)^n \right] \tag{3.15}$$

where $q = \ln v + \mu/a$.

Expanding this expression for n = 2, 3, and 4 gives

$$E(y - \mu)^2 = a^2[\Gamma''(v)/\Gamma(v) - 2q\Gamma'(v)/\Gamma(v) + q^2], \tag{3.16}$$

$$E(y - \mu)^3 = a^3[\Gamma'''(v)/\Gamma(v) - 3q\Gamma''(v)/\Gamma(v) + 3q^2\Gamma'(v)/\Gamma(v) - q^3], \tag{3.17}$$

and

$$\begin{aligned} E(y - \mu)^4 = a^4[&\Gamma^{(4)}(v)/\Gamma(v) - 4q\Gamma'''(v)/\Gamma(v) + 6q^2\Gamma''(v)/\Gamma(v) \\ &-4q^3\Gamma'(v)/\Gamma(v) + q^4]. \end{aligned} \tag{3.18}$$

The derivatives $\Gamma^{(n)}(v)$ can be expressed in terms of polygamma functions. A fairly thorough account of these functions is given in A&S (p 258-260) with tables for $\psi^{(n)}(v)$, n = 0, 1, 2, and 3 beginning on p 267. The polygamma functions are defined as

$$\psi^{(n)}(v) = d^n \psi(v)/dv^n \qquad 3.19$$

where

$$\psi(v) = d \ln \Gamma(v)/dv \qquad 3.20$$

so that

$$\Gamma'(v) = \Gamma(v)\psi(v). \qquad 3.21$$

In the following expressions for the derivatives $\Gamma^{(n)}(v)$ the v dependence is suppressed. From $\Gamma' = \Gamma\psi$,

$$\Gamma''(v) = d\Gamma'/dv = \Gamma'\psi + \Gamma\psi' = \Gamma(\psi^2 + \psi') \qquad 3.22$$

$$\Gamma'''(v) = d\Gamma''/dv = \Gamma'(\psi^2 + \psi') + \Gamma(2\psi\psi' + \psi'') \qquad 3.23$$

$$= \Gamma(\psi^3 + 3\psi\psi' + \psi'')$$

$$\Gamma^{(4)}(v) = d\Gamma'''/dv = \Gamma'(\psi^3 + 3\psi\psi' + \psi'') + \Gamma(3\psi^2\psi' + 3(\psi')^2 + 3\psi\psi'' + \psi''')$$

$$= \Gamma(\psi^4 + 6\psi^2\psi' + 4\psi\psi'' + 3(\psi')^2 + \psi''') \qquad 3.24$$

Substituting into the central moment expressions given above and collecting terms yields

$$E(y - \mu)^2 = a^2 \psi'(v) \qquad 3.25$$

$$E(y - \mu)^3 = a^3 \psi''(v) \qquad 3.26$$

and

$$E(y - \mu)^4 = a^4 [3(\psi'(v))^2 + \psi'''(v)]. \qquad 3.27$$

The second central moment, or variance, for v = 1 is simply

$$E(y - \mu)^2 \equiv \sigma_y{}^2 = a^2 \psi'(1). \qquad 3.28$$

From eq 6.4.2 and 23.2.24 of A&S

$$\psi'(1) = \zeta(2) = \frac{\pi^2}{6}, \qquad 3.29$$

so that

$$\sigma_y \cong (4.343)(1.283) \cong 5.57, \qquad 3.30$$

in agreement with well-known results (see Dyer [1970]).

The coefficients of skewness and kurtosis, α_3 and α_4, are given by

$$\alpha_3 = -\psi''(v)/[\psi'(v)]^{3/2} \qquad 3.31$$

and

$$\alpha_4 = 3 + \psi'''(v)/[\psi'(v)]^2 \qquad 3.32$$

Asymptotic expressions for ψ', ψ'' and ψ''' are given on p 260 of A&S and are (including the first two terms only)

$$\psi'(v) \sim \frac{1}{v} + \frac{1}{2v^2} + \ldots \qquad \text{(Eq 6.4.12, A\&S)}$$

$$\psi''(v) \sim -\frac{1}{v^2} - \frac{1}{v^3} - \ldots \qquad \text{(Eq 6.4.13, A\&S)}$$

$$\psi'''(v) \sim \frac{2}{v^3} + \frac{3}{v^4} + \ldots \qquad \text{(Eq 6.4.14, A\&S)}$$

Thus to lowest order in v

$$\alpha_3(v) \sim v^{-1/2} \to 0 \quad \text{as } v \to \infty \qquad 3.33$$

and

$$\alpha_4(v) \sim 3 + v^{-1} \to 3 \quad \text{as } v \to \infty, \qquad 3.34$$

and hence these coefficients are asymptotically Gaussian.

3.4 Some Properties of the Log-Gamma Distribution

3.4.1 Density Curves

The log-gamma pdf illustrated in figure 7 for v = .05(.25)2.5, exhibits a pronounced skew, as does the gamma pdf. Under the negative log transformation

applied here, the skew is to the right of the mean. As v increases the skewness becomes much less pronounced. A feature resulting from transforming about $Ex = v\lambda$ is the invariance of the mode with respect to v. That is, let $h(y) = v(y/a - e^{y/a})$ and let $C_v = v^v/|a|\Gamma(v)$, then

$$f(y) = C_v e^{h(y)}$$

and hence

$$f'(y) = (v/a)(1 - e^{y/a})C_v\, e^{h(y)}$$

Setting $f'(y)$ to zero yields the most probable value of y, the mode, as

$$\hat{y} = a\, \ell n\, 1 = 0$$

for all v.

The second derivative takes the form

$$f''(y) = a^{-2}[-ve^{y/a} + v^2(1 - e^{y/a})^2]C_v e^{h(y)}$$

and when set equal to zero yields a pair of inflection points for each v, say

$$y_{\pm} = a\, \ell n[1 + \frac{1 \mp \sqrt{4v + 1}}{2v}]. \qquad 3.35$$

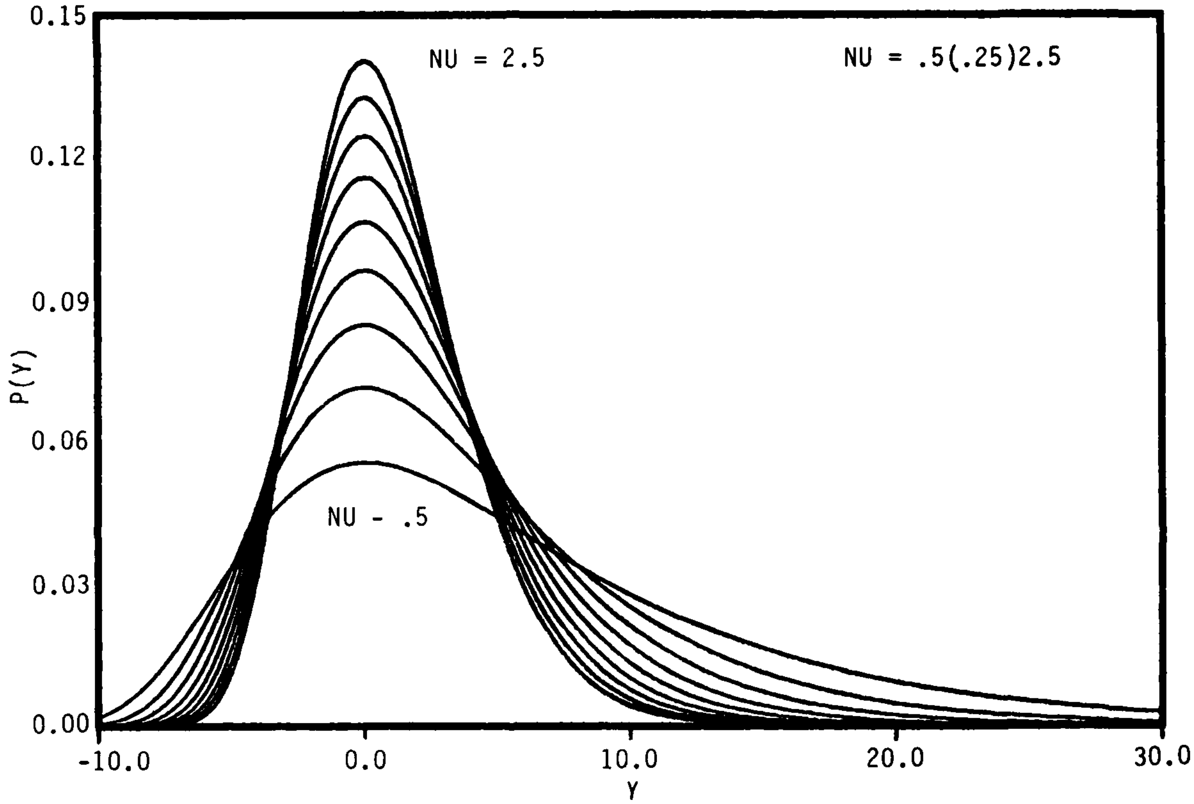

Figure 7. Log-gamma density curves.

As v increases the distance between the corresponding pair of inflection points decreases. Thus the distance between inflection points could serve as a measure of peakedness.

3.4.2 Log-Gamma CDF

Given that x has the distribution $G_v(x)$ (eq 2.28), then under the transformation $y = -|a|\ell n(x/v\lambda)$ the distribution of y, say $F_v(y_o)$, is given by

$$
\begin{aligned}
F_v(y_o) &= P_r(y \le y_o) = P_r(-|a|\ell n(x/v\lambda) \le y_o) \\
&= P_r(x > v\lambda e^{-y_o/|a|}) \\
&= 1 - P_r(x \le v\lambda e^{-y_o/|a|}) \\
&= 1 - G_v(v\lambda e^{-y_o/|a|}) \\
&= 1 - P(v, ve^{-y_o/|a|}), \qquad 3.36
\end{aligned}
$$

where P(v,x) is discussed in section 2.3.3.

Plots of $F_v(y)$ are presented in figure 8 for v = .75(.25)2.25.

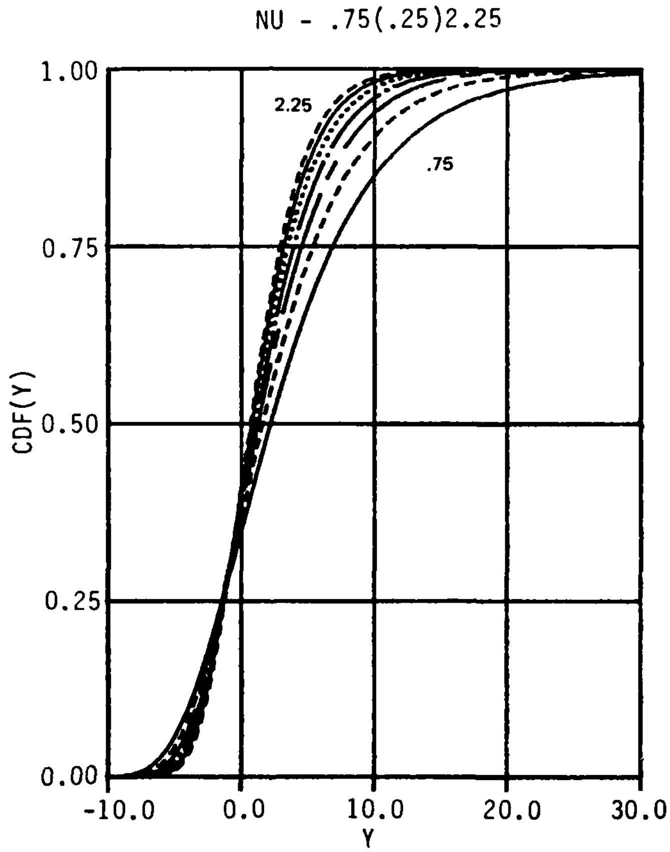

Figure 8. Log-gamma CDF curves.

3.5 Parameter Estimation

By taking the transformation about the expected mean intensity, $E(x) = \nu\lambda$, only the parameter ν needs to be estimated from samples in dB-space. The parameter λ is implicitly determined through the calculation of $\bar{x}$ in intensity space. In practice λ need not be known explicitly. Since $y = a\,\ell n(x/\nu\lambda) = a\,\ell nx - a\,\ell n\nu\lambda$ then denoting propagation loss measured at range r_n by H_n, the corresponding sample point is $y_n = H_n - \langle H\rangle$ where $\langle H\rangle = -10 \log \bar{x}$ and $\bar{x} = \Sigma x_n/N$. The y_n from a specified range interval represent deviations of loss from the interval mean.

Assuming that the y_n are independent and identically distributed (iid)*, the maximum likelihood estimator of ν can be determined from (see section 2.1.1)

$$\ell n\nu - \psi(\nu) = -\bar{y}/|a|, \qquad 3.37$$

where

$$\bar{y} = \Sigma y_n/N = \bar{H} - \langle H\rangle. \qquad 3.38$$

Figure 9 presents $\bar{y}$ versus ν, providing a graphical means to estimate ν given $\bar{y}$. The dashed curve gives σ versus ν and can be used as a consistency check for ν obtained from $\bar{y}$, since $\sigma^2 = N^{-1}\Sigma(y_n - \bar{y})^2$ is an estimate of $E(y-\mu)^2 = a^2\psi'(\nu)$.

The following examples illustrate the maximum likelihood parameter estimation method. The first example is taken from a bottom-limited data set analyzed by Pederson [1980]. The sample size is a rather awesome 5,398 points, which perhaps accounts for the quality of fit that is evident in figure 10.

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* Obtaining iid sample points is evidently an ancient artform handed down from one generation of data reduction specialists to another, and involves closely guarded secret black arts such as detrending, decimation, and "proper" selection of interval length.

The impressive sample size was obtained by combining range-detrended data sets derived from several independent measurement events. The curve over-plotted on the histogram is a log-gamma density function corresponding to $v = 1.12$. The appropriate value of v can be obtained graphically from figure 9 or by iteratively solving equation 3.37 for v given that $\bar{y} = 2.22$.

An example based on model-generated data is demonstrated by the histogram of figure 11. A subsample consisting of 250 points generated by RAYMODE-X was range-detrended by subtracting the incoherent outputs from the coherent outputs. The sample mean is 1.7; thus from figure 9 the estimated value of v is 1.3. The closeness of fit is not particularly impressive near the mean, but the degree of coincidence is certainly acceptable along the tails.

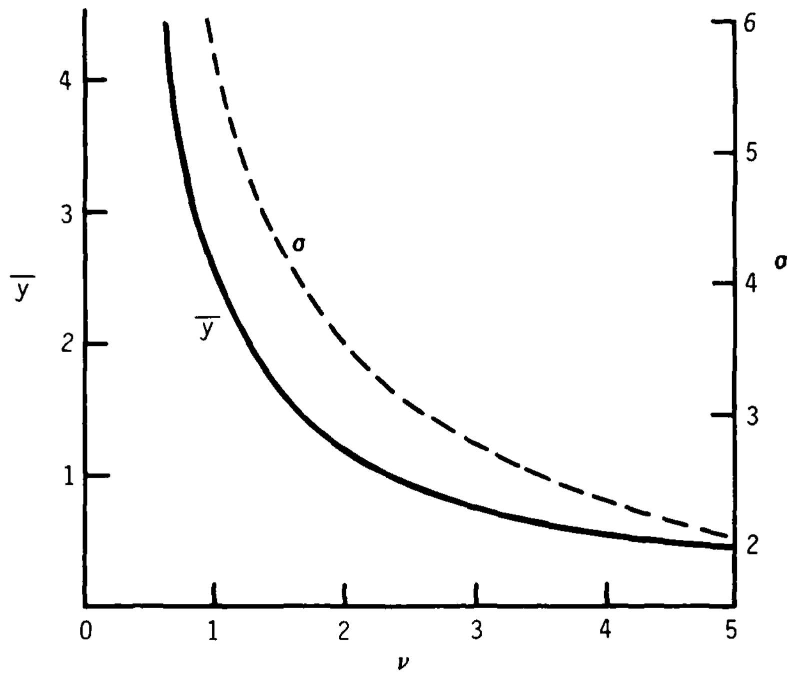

Figure 9. Log-gamma parameter estimation curves.

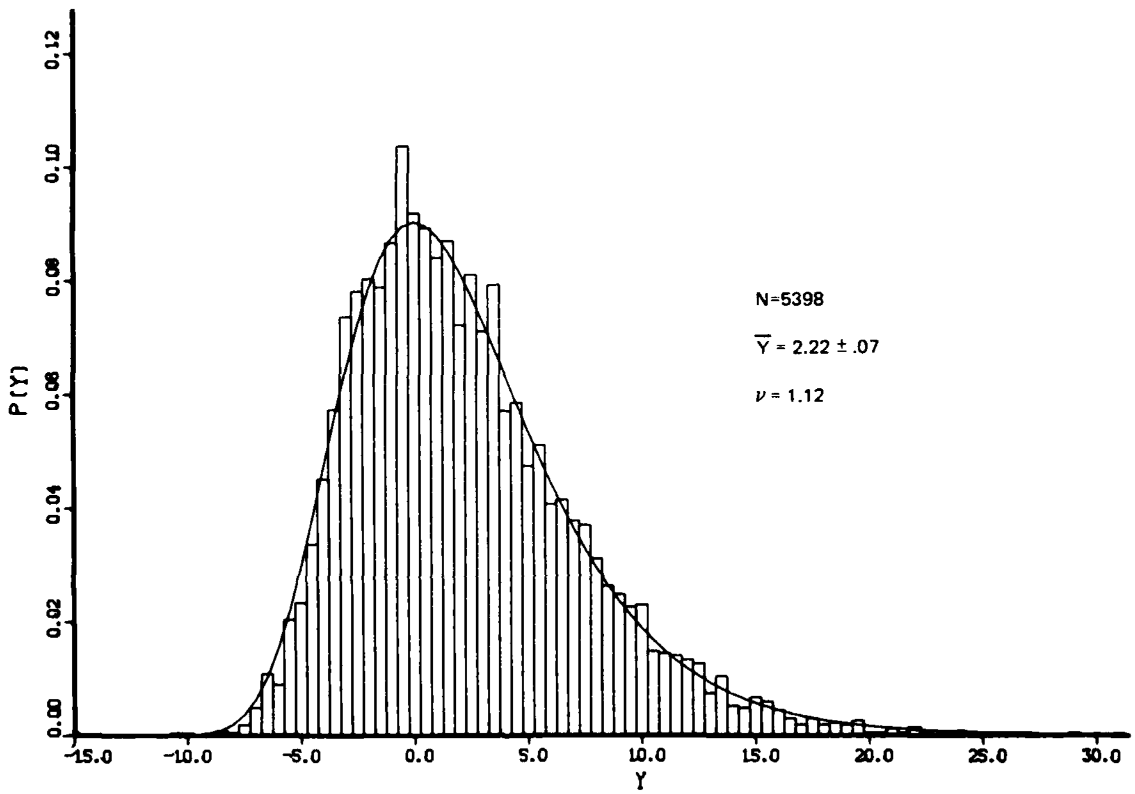

Figure 10. Histogram of bottom-limited data.

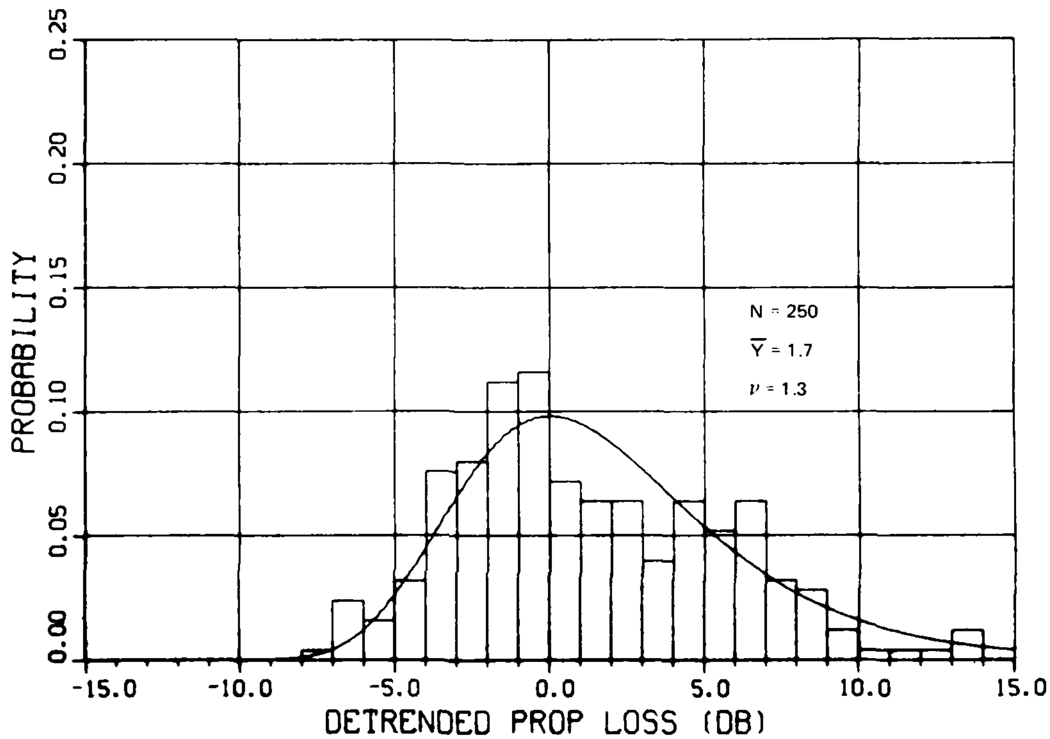

Figure 11. Histogram of propagation loss data generated by raymode X.

4. RECOMMENDATIONS FOR FUTURE WORK

4.1 Extensions to AMEC Software

The log-gamma and Gaussian "graphical aids" are included in the AMEC software package largely as a result of efforts in support of a limited propagation data assessment project recently conducted by Pedersen [1980]. Initial data snooping clearly indicated that most of the collected data derives from distributions exhibiting skewed tails. As a consequence there was no pressing need to consider other distribution possibilities. Log-gamma and Gaussian distributions do not always provide adequate fits, however, as demonstrated by Frisk [1978] in his comparison of exponential, gamma and log-normal distributions against low-frequency, long-range propagation data. At the lowest of three frequencies (9.8, 110, 262 Hz) all of the distributions produced an acceptable fit. At 110 Hz only the log-normal with σ = 5.6 dB was adequate, and at 262 Hz none was acceptable.

Frisk suggests that the Rice distribution may yield a reasonable fit to data when log-normal and gamma distributions fail. Derivations of the Rice pdf are given in Rice [1954], Davenport and Root [1958], Whalen [1971] and Ol'shevskii [1967]. The Rice pdf may be put in the form

$$p(v;a) = ve^{-1/2(v^2 + a^2)} I_0(av)$$

where v and a are ratios of the envelope and the sine wave amplitude to RMS noise. Urick [1977] examines this distribution in terms of a randomicity parameter T defined as

$$T = \frac{2}{a^2 + 2} .$$

When $a \to 0$, $T \to 1$ and $p(v;a)$ reduces to the Rayleigh pdf, and when both v and a are large $p(v;a)$ exhibits Gaussian behavior. Since $T \to 0$ as $a \to \infty$ then T ranges between 0 and 1. As an estimate of T, Urick uses the value that yields the closest agreement between the Rice CDF and the sample CDF. This approach appears to be more efficient (computationally) than the maximum likelihood procedure. That is, reverting back to the parameter "a" (for notational convenience), the log-likelihood function for a sample of size N is

$$\ell n\, L(a) = \ell n \prod_{n=1}^{N} v_n - \frac{1}{2} \Sigma\, v_n^2 - Na^2/2 + \Sigma\, \ell n\, I_o\, (av_n).$$

Setting $d\, \ell n\, L(a)/da$ to zero yields

$$\hat{a} = \frac{1}{N} \Sigma\, v_n\, I_1\, (\hat{a}\, v_n)/I_o(\hat{a}v_n)$$

which must be solved iteratively--a discouraging calculation to contemplate. A transformation of the form $u = v^2$ results in a non-central chi-square distribution (p. 113, Whalen [1971]) for which extensive tables and computational algorithms have been developed (Harter and Owen [1973]). For AMEC purposes, however, confining T (or equivalently a non-centrality parameter) to a few selected values appears to offer the most efficient procedure.

The Rice distribution provides a connecting link between the Rayleigh distribution ("slow" fluctuations) and the log-normal distribution ("fast" fluctuations). Delineating the space of fluctuations vertically with a "phase-fluctuation strength" parameter Φ and horizontally with a "diffraction" parameter Λ, Flatte et al [1977] suggest the Rice distribution to characterize amplitude fluctuations along the fuzzy border separating saturated ($\Phi^2\Lambda >> 1$) and unsaturated ($\Phi^2\Lambda << 1$) regions. Thus as a "transition curve" the Rice distribution affords an alternative to "arbitrary" Pearson and Edgeworth

frequency curves (see, for example, Elderton and Johnson [1969]). Part of the impetus for characterizing amplitude fluctuations with "standard" distributions vice "best-fit" frequency curves stems from the desire to have a "physics-based" framework for calculating bounds on model predictions.

4.2 Prediction Bounds

Serious attempts to place confidence bounds on propagation loss predictions have been avoided because the dependence of fluctuations on model input parameters is not well enough understood. Moreover, the amount of information contained in a given set of model inputs might not be sufficient. Eclipsing such pessimism, however, the assumption is made that RMS fluctuations can be related in some way to model input parameters.

Three approaches to this problem are considered here. The first approach, and the most expedient, assumes that the deviation of propagation loss from its mean value follows a $N(0,\sigma^2)$ law with $\sigma = 5.6$ dB. Using a second-pass scheme, the predicted loss versus range curve is divided into range intervals according to predominant path type. For each such interval the 90% confidence limits are given by $\hat{H}_n \pm 1.96\sigma/\sqrt{N}$, where N is obtained by dividing the decorrelation length into the range interval length. A slight variation to this approach entails substituting a more "appropriate" distribution for the normal law, such as the log-exponential, the log-gamma, or the log-Rice distribution.

The other two approaches are by comparison much more extensive in scope in that each requires considerable data analysis effort. In one of these approaches, the dependence of RMS fluctuations on environmental-acoustic (EVA) parameters is determined using regression methods. That is, the RMS fluctuation σ_r at range r is regressed on selected EVA parameters (or functions of EVA parameters).

The third (and most elaborate) approach applies Baysian methods (see for example p 234-7 of Lindgren [1978]). Let $\bar{y}$ denote the deviation of predicted loss $\hat{H}$ from its mean $\langle\hat{H}\rangle$ over a specified range interval, and let $\bar{w}$ denote the deviation of measured loss H from its mean $\langle H\rangle$. Suppose that $\bar{w}$ is found to have a distribution with pdf $p(\bar{w}|\Theta)$ conditioned on a known value of the parameter Θ. The parameter Θ can take the form of an n-tuple, although in practice it typically is either a single parameter or a pair of parameters. For example if $p(\bar{w}|\Theta)$ is log-exponential then $\Theta = \lambda$, or if $p(\bar{w}|\Theta)$ is log-gamma then $\Theta = (\nu,\lambda)$, or if $p(\bar{w}|\Theta)$ is Gaussian then $\Theta = (\mu,\sigma^2)$. During a given measurement exercise information about Θ is gained, yielding a prior pdf for Θ, say $g(\Theta)$. A posterior density function $h(\Theta|\bar{w})$ is then given by

$$h(\Theta|\bar{w}) = g(\Theta)p(\bar{w}|\Theta)/p(\bar{w})$$

where $p(\bar{w}) = \int g(\Theta)p(\bar{w}|\Theta)d\Theta$. Finally the predictive density function $f(\bar{y}|\bar{w})$ is determined from

$$\begin{aligned} f(\bar{y}|\bar{w}) &= \int f(\bar{y}|\Theta)h(\Theta|\bar{w})d\Theta \\ &= \frac{\int f(\bar{y}|\Theta)p(\bar{w}|\Theta)g(\Theta)d\Theta}{\int p(w|\Theta)g(\Theta)d\Theta} \end{aligned}$$

The pdf $f(\bar{y}|\bar{w})$ along with its estimated parameters can then be used in calculating confidence limits. Note that both the regression approach and the Baysian approach tend to be "EVA-specific" since in each case the final results are tied to a particular set of environmental-acoustic conditions.

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